SEEING -ACTING - SPEAKING IN GEOMETRY: A CASE STUDY

Thomas Barrier, Christophe Hache, Anne-Cécile Mathé

SEEING–ACTING–SPEAKING IN GEOMETRY: A CASE STUDY

Thomas Barrier[1], Christophe Hache[2], & Anne-Cécile Mathé[1]

[1]LML – Université d'Artois, France

[2]LDAR, Université Paris 7, France

The purpose of this paper is to describe and analyse the confrontation and changing processes of frequentation modes (seeing – acting – speaking) of 1st grade secondary school students (10-11 years old) during two geometric construction tasks. This work is based on a logic analysis of the mathematical concepts involved: the midpoint of a line segment and the circle.

INTRODUCTION

When we try to better understand how the students' "relationship to knowledge" changes in a learning situation in geometry, language interactions are often considered as preferential indicators of the reference framework underlying the students' geometry activity. In this case, the language remains subordinated to the analysis of the physical activity. As for us, wee try to consider language not only as a reflection of the conception of an acting subject but to take into account, in our analysis, both the language and the physical part of geometry activity. For this purpose, Bulf, Mathé, and Mithalal (2011) introduce the notion of *frequentation mode* which aims at accounting for the consistency of physical and language dimensions of activities in geometry. They insist on the fact that, on the one hand, language practices are not different from other practices, they have no hierarchy (in order to avoid language issues being pushed into the background of the action, especially in geometry) and, on the other hand, language practices are constituent parts of subjects' knowledge rather than mere reflections of pre-existent knowledge.

Frequentation modes of a geometrical object should articulate three dimensions at the same time: a way of *seeing* in geometry, which will form most of our theoretical horizon and will allow to make students' appreciation of a geometry activity more clear; the ways of *acting* taken by the subject and the instrument usage rules; the ways of the *speaking* and the subject's discourse on objects and actions. The word *seeing*, that we prefer here to the word *thinking,* even if they have analogous meanings, underlines the importance of visualization in geometry (Duval, 2005; Duval & Godin, 2006). As for the word *acting*, we have decided to focus specifically on the artefacts as a didactical variable which is determining in construction situations (Perrin-Glorian, Mathé, & Leclerc, 2013). At last, as for the word *speaking,* we shall intend to describe the specific *language games* (Wittgenstein, 1953) implemented by students and their changing process (Barrier, 2011; Mathé, 2012). Therefore, defining a frequentation mode consists of describing a geometry activity by a way of thinking and considering modalities of action consistent with a certain discourse characterized by its structure and the meaning given to the terms by the subject in the context. In this context, learning is described as a changing process of frequentation modes to a relationship in line with knowledge objects.

This approach seems to echo the hypothesis of a "discursive community" (Bernié, 2002; Jaubert, Rebière, & Bernié, 2003), a French approach analogous to the discursive or communicational one (Kieran, Forman, & Sfard, 2001). In a Vygotskian perspective, these authors introduce the notion of a school disciplinary discursive community: knowledge is tightly linked with the community which created it, learning in a school subject area is learning how to act-think-speak somewhat like experts. This means learning how to take position in a social universe characterized by interesting issues, specific objects, material and practices, in particular language practices, learning *Discourse practices* (Moschkovich, 2007).

However, we do not have the feeling to follow a radical *communicational* way of thinking (Sfard, 2001). Indeed, we are as interested in the dialogical and social aspects of discourse as in the semantic dimension of language, i.e. its ability to refer to external mathematical objects. The purpose of this paper is to show, from a case study, how a logic analysis of concepts can help to describe frequentation modes and their dynamics. We shall use the language of the first-order logic, i.e. the fundamental logic categories will be those of object, predicate and relation. This work is part of a larger project aiming at creating tools that could account for the geometrical practices both in the physical and language dimensions of these practices.

CORPUS AND METHODOLOGY

The extracts analyzed in the following pages are taken from data collected for another work dedicated to issues relating to the use of history of mathematics in the classroom. Detailed information on the context of data collection and on students' tasks is available in Barrier, de Vittori, and Mathé (2012). In this study, we selected some sequences of a lesson during which 1st grade students (10-11 years old) in a French secondary school had to draw a square on the playground using unusual artefacts: a chalk and a rope. The construction program given to the students and the expected figure are available in annex. The lesson lasts just under one hour and takes place outdoors, in the playground. Students are divided in groups of three or four and the teacher moves from one group to the other. The sequences reported have been filmed.

In this problem, students are asked to construct a midpoint and a circle. Starting from the logic analysis the concepts of circle, we shall attempt to bring out *a priori* the potential of frequentation modes which could be considered for the geometrical objects involved in the problem set. This will provide an analysis framework which will be used, during the progress of the sequence, to identify coexisting divergent interpretations of the situation and will help to better understand how they evolve. We will observe the students' gestures and procedures as well as the dialogues between them and with the teacher.

CONSTRUCTING THE MIDPOINT

A priori analysis

A line segment may be seen in different ways: it may be seen as a part of a straight line or as a pair of points. In the first case, the midpoint of a line segment is characterized by a binary relation between two objects, a geometrical object of dimension 1 – the line segment – and an object of dimension 0 – the midpoint (example of statement of the relation: the midpoint is the point of the line segment that splits it into two parts of equal lengths). In the second case, it can be characterized by a ternary relation between three objects of dimension 0: the end points of the line segment and the midpoint (example of statement of the relation: the midpoint is the point aligned and equidistant with the two end points). These two ways of seeing call for objects which differ in number and nature (in dimension). From a physical action standpoint, there are many possible construction procedures. In the first grades of secondary school, the more usual method involves using a graduated ruler (to measure the length of the line segment and to plot half of the length from one of the ends). The property explicitly studied in this procedure is the fact that the midpoint M of a line segment $[AB]$ fits the equality $AM = ½ AB$. Questions such as the alignment of the midpoint with the end points or the midpoint belonging to the line segment are hidden by using the graduated ruler. Another method will be used later: plotting the perpendicular bisector using a ruler and a compass. In the present situation, the rope can be used as an artefact to split the length and check the alignment. The construction requires to explicitly account for both alignment and equidistance properties. It should be noted that the construction program proposed (annex) does not explicitly require plotting the line segment involved. There are several procedures available. We shall describe three of them:

P1. The first procedure consists of plotting the line segment on the ground by stretching out the rope, identifying the ends if necessary and then folding the rope in two equal parts. The midpoint is obtained by plotting the length from one of the ends. In this case, the property under which the midpoint belongs to the line segment is no more explicit than in the ruler procedure, since the question of alignment (or of belonging to the line segment) is evened out by the plotting of the line segment that is made independently of the construction of the midpoint.

P2. If the line segment has not been previously plotted, the rope can be laid in a straight line on the ground and then be folded in two parts by moving only one of its ends, the other end staying in the same place. Therefore, the midpoint is placed at the end of the new line segment thus obtained. Theoretically, this procedure does not call for the issues about alignment or belonging to the line segment, but the question arises from a practical standpoint, since it is difficult to move half of the rope while the other end stays still.

P3. The third procedure explicitly accounts for the alignment property. It starts by laying the rope on the ground as described in the procedure P2 and by identifying the ends. Then the rope is folded in two equal parts and the new length is used to plot an arc of a circle, the centre of which is one of the ends of the line segment. The midpoint is obtained by determining, using the rope, the point on this arc which is aligned with the identified ends (the line segment can be plotted or not).

We consider that the distinction made between the different conceptions of the notion of midpoint, in terms of binary or ternary relations, in addition to the *a priori* analysis of possible procedures, may contribute to demonstrate the potential of frequentation modes which could be considered for the midpoint (of a line segment) object. The following *a posteriori* analysis should identify the frequentation modes in which the students stand, detect the possible coexistence of different frequentation modes and intend to better understand how the change towards a frequentation mode of the midpoint notion in line with the school expectations at this educational level operates.

***A posteriori* analysis**

In this paper, we shall focus on the physical and dialogical practices of a group of three students (E1, E2 and E3) and their interactions with the teacher (H) about the construction of the midpoint of line segment [*OE*]. The three points *O*, *E* and *I* are identified by a cross and by their corresponding letter on the playground. The line segment [*OE*] is not plotted and the three points *O*, *E* and *I* do not seem to be aligned. Then, the teacher intervenes and asks the students to explain how they have proceeded. E1 "shows" the way they used to construct the point. He starts by joining the two ends of the rope, joining both his hands to openly show the half-length obtained. This means that the group perceives the length constraints imposed on the construction of the midpoint. Then he lays a part of the rope on the ground. One end of the rope is placed in *O*, while the other end stays in E1's hand and the rope is laid so that it passes by *O* and by *I* (but it does not pass by *E*). Now, he folds the end he holds towards point *O*, without exerting any other pressure on the rope. It seems that this group has used procedure P2, in a more or less successful manner. The interactions proceed in the following way:

H:	No, no, but ... have we got a means to check if it is the midpoint?
E1:	Why, yes
H:	What could we do?
E1:	Plot a line.
H:	A line? […] Check there... How, how did you place your rope to check that this is the midpoint? [Pause] How will you proceed?
E3:	We lay it/ [E3 points his finger towards *O*]
H:	Therefore, we put one end here and then the other end/ [E1 puts an end of the rope in *O*]
E3:	We fold it
H:	Yes, and the other end? You must stand...
E1:	We fold it like that [E1 follows the procedure described above]

H: Well, this is not what I want

We can assume that H checks the control procedure which consists of using the rope as an artefact to check that the three points *E*, *O* and *I* are aligned. As for the students, they seem to be in a frequentation mode that is definitely different from the notion of midpoint. They focus on the distance constraints and they only consider global perceptive retroactions which cannot invalidate their construction strategy. This misunderstanding appears in the form of language interactions. For example, when H says "and then the other end" then goes on with "Yes, and the other end ", it seems that he expects an answer with something like a specification of the position where the other end of the rope should be placed. The students' answers are in the action field, to fold the rope in a certain manner, rather than in the place field. This extract shows how language interactions may be a place of confrontation between conflicting frequentation modes and an (attempt of) negotiation towards a shared frequentation mode. For example, when the teacher repeats E1's statement "we lay it there" by saying "Therefore, we put one end here and then the other end", he attempts to direct the students' look towards the ends of the line segment and introduces the end *E* as a reference of a position. Thus, he intends to lead the students towards an interpretation of the "midpoint" object defined by a ternary relation between three points. The technique implemented by the teacher aims at pointing out that point *E* should be taken into consideration. The purpose is to set a shared objects field from which construction language games could be compared. Nevertheless, students are not able to put on language indicators used by the teacher and they do not recognize the specific form of language game he wants them to play. Finally, this misunderstanding leads the teacher to artificially put aside the strategy of this group. Therefore he decides to introduce by himself the third point necessary to make them shift to the punctual standpoint:

H: [H puts his forefinger on point E] Yes but here, in relation to this point, is there a means to check that your point placed there will be the middle point, the midpoint of your line segment?

E3: Well, we plot er... the rope.

However, it is difficult for students to use the rope as a geometrical artefact to plot straight lines. So far, the rope has been laid on the ground in an approximate straight line, without exerting special pressure on its ends. Now, the teacher takes over a more important part of the problem. He uses the language to simultaneously set in action the three points the alignment of which is to be questioned and clarifies the fact that they must be linked by a specific relation.

H: If your point is the midpoint, how should these three points be?

E1: On the same straight line

H: On the same straight line, well then have we got a means to check your three points are really on the same line? What can we do?

E1: Oh no, there are like this! [E1 shows that the points are not aligned]

H: Well, how can we check then, how can we be sure it will be placed correctly? […] You cannot see how we can check the points alignment?

E3: Er… no

H: Well, your task will be... You have to find the way, just think, sort it out yourselves, find how to check that your three points are correctly aligned, that's all [H goes to another group].

This time the students see the necessity to align the points, thus focusing on the line segment, to the exclusive consideration of the lines and lengths by a punctual look, inducing a possible questioning on the points alignment.

In this first example, we have tried to point out the consistency between the modalities of physical action, the discourse and the way of looking at the figure for a given group of students, even when this consistency is disturbed by the teacher's intervention. Let us now present a second example.

PLOTTING A CIRCLE

A priori analysis

From a logical and mathematical standpoint, the circle can mainly be seen (for other characterizations of the circle, cf. Artigue & Robinet, 1982) as:

- a set of points characterized by a relation: the fact that they are at a given distance (radius) from a given point (centre). This representation corresponds to a plot using a compass or a rope, but also a "point-by-point" plotting (plotting multiple points at a given distance from the centre, then plotting a line if necessary, or linked line segments)
- a continuous line with constant curving. This vision of a circle is hardly operational except for freehand plotting (it can be combined with plotting a few points or few diameters then applying the point-by-point plotting described above), this characterization can also be used for checking a freehand plotting (or using artefacts if necessary)
- the given length line which contains the largest surface area (not quite operational, somewhat corresponding to the circle of a children's dance)
- a line with infinite number of axes of symmetry (not quite operational but it can be used to check during plotting)

It should be noted that the first characterization calls for a relation between objects of dimension 0 (points, including the midpoint which is "exterior" to the graph) whereas the three other points use properties applicable to a single object of dimension 1 (the line).

Of course, the rope can be used to plot circles (or arcs of a circle). This construction requires to follow the same preliminary steps as for plotting with a compass (this artefact is almost always used for plotting circles in a classroom): decision of the radius length to be used (if necessary, selection of the line segment to be plotted, or modalities of length measurement if the length is given with a numerical value) and of the centre around which the circle must be plotted. If the rope is used to plot, it might be difficult to hold one of its ends in a fixed point during rotation. With a compass, when the space between the legs has been fixed, the equidistance property of the points of the circle, or from the line to the centre, is accounted for by the stiffness of the artefact itself. In the context of plotting with the rope, this property is tightly linked with the fact that the rope must be held tight during the whole plotting process. It is physically felt by the student who makes the plot of the circle and it is the required condition to plot the circle. Plotting with a rope usually corresponds to plotting in the meso-space, whereas the compass is commonly used in the micro-space of the sheet of paper. This parameter induces different gestures: plotting with a compass requires hand work, while plotting with a rope requires moving the body and the arms and sometimes the intervention of two students is needed (one student keeps one end of the rope on the centre of the circle whereas the other holds the tight rope and draws the required circle with the other end).

In this case, the links between characterizations and modalities of construction clearly appear, as well as the links between the characterization and the nature of the circle object. The objects explicitly or implicitly handled and the nature of their relations (binary relation, property, etc.) differs from one characterization to the other. What is considered: the centre? the radius? Is the circle seen as a set of points? a line?

A posteriori analysis

In this part, we shall focus on the analysis of the sequence with the work of a second group, again made up of three students. The students are going to plot the circle with the line segment [OE] as a diameter.

The analysis of language interactions shows us that without their usual artefacts the students cannot instinctively adopt a mathematical frequentation mode. They suggest to draw "a round shape" and even a "normal round shape", i.e. a round shape which does not refer to mathematics but rather to what they usually use outside the specific approach of spatial issues raised in geometry. This justifies freehand plotting here. The standpoint on the circle adopted is that of a rounded shape made of a line (a closed line, characterized by its constant curvature and/or its symmetries for example). This is a global rather than a local point of view, since it calls for lines and not for points (and the centre of the circle is not evoked). The teacher's interventions can be seen as attempts to (re)-position the students in a frequentation mode of the circle object which is more in line with the school mathematical expectations, to guide them to express their practical concerns through a mathematical questioning of the properties of the objects and artefacts involved, using a language suitable for the school mathematical context. The background movement that the teacher attempts to bring about is stimulated by the questions: "A circle, what is it? What is a circle? ". Raised by the teacher, these questions call for a change in the students' way of seeing. The questions on the nature of the circle are not only or mainly aimed at obtaining a definition of the circle in return. The objective is to lead the students to "see" the circle as it is usually seen in the school mathematics context. Some of the answers given by the students may seem to be tautological ("A circle", " It's a circle", "Well, it's a circle") and useless in the context of knowledge. The students seem to be aware of it, but it is not the case if they are analyzed considering how the dialogue works and how the practices are inserted in the required context. These language interactions show a change in position. This movement is also revealed by the fact that, in other following answers, terms which are specific to mathematical vocabulary are introduced (centre, diameter and radius).

With the term "the-circle-with-a-centre-X-and-a-radius-Y" (used twice in an identical manner) and the explicit allusion to the «definition» of the circle, the teacher clearly introduces a formal dimension (Hache, in press), above all in relation to the situation of the exercise and the supposed frequentation mode of students. Besides, from the knowledge standpoint, the teacher, by using the words "centres" and "radius" for example, refers to the prevailing definition of the circle in the school context, i.e. the circle seen as a set of points placed at a same distance of a given point. As already mentioned, this characterization calls for a property which is made natural by using a compass in the usual situations of plotting. On the other hand, it differs from the instinctive characterization adopted so far by the students, which rather seemed to lie on the circle as a line, characterized by its constant curvature. In practice, shifting from one conception to another is not instinctive (Artigue & Robinet, p. 49), all the more as the characterizations appeal to students to have different looks on the figures (Duval & Godin, 2006). The last teacher's intervention in the above extract can be understood as an inducement to fit the way of seeing ("therefore, is formed by what?") with the way of speaking ("the circle with a centre O and a 3 cm radius").

CONCLUSION

The purpose of this paper was to describe and analyse the confrontation and changing processes of frequentation modes of 1st grade students (10-11 years old) during two geometric construction tasks. This work was based on a logic analysis of the mathematical concepts involved: the midpoint of a line segment and the circle. In both cases, according to the adopted standpoint, the mathematical concepts can be described from different categories of logic (property, binary or ternary relation) on objects different in number and nature. This analysis, although it was quite brief, seemed to be useful to consider the consistency and practical harmonization of the three dimensions "seeing – acting – speaking" we called for to describe the frequentation modes (Moschkovich, 2007, would maybe have said a *Discourse*). We could thus observe that the change in the way of seeing ("Oh no, there are like this!") of the students who worked on the construction of the midpoint was produced by the teacher's language action aimed at setting the objects *ends of the line segment* as references for some words in the language games specific to the geometry practice at school. This change in their way of seeing is associated with new possible uses of the artefact *rope* (physical dimension of geometry practice). As for the plotting of the circle, we attempted to show how the language practices could be linked with the plotting methods. It seems that the expression "a normal round shape" can be related to extra-school practices which justify the "freehand" plotting that we compared with a global vision of the circle as a rounded shape. If this approach is somewhat justified, the related vision is not that in use in mathematics at secondary school. On the contrary, the language game which calls for the expression "the circle with a centre O and a 3cm radius", and which is initiated by the teacher, introduces some elements required for invoking a punctual standpoint on the circle, in particular the centre of the circle. This centre, exterior to the line actually plotted, must be taken into consideration to implement the techniques which call for the equidistance relation.

We are just at the start of our research and we are not sure to be able to offer a pertinent discussion of these results. Nevertheless, we will try to situate it inside the today well-established *discursive framework* in mathematics education (Sfard, 2013). Our feeling is that the former semantic and dialogic perspective could be one way to consider both social and external aspects of language. Mathematical language games could be *outdoor games* (Hintikka, 1996), i.e. games involving the objects of the language one speaks. Analysis of communication quite often emphasizes interpersonal interactions. In this work, we think it necessary to integrate a specific focus on the interaction between students and external (even if dialogically constructed) mathematical objects, with the help of a logic analysis of the concepts at stake. Of course, all of this is nothing new. For instance, the Theory of Didactical Situation tradition in France has a long time ago pointed out the educational interest of the students-*milieu* interactions (Brousseau, 1997) and inside the communicational approach, Sfard (2001) clearly accounts for the “object-level aspects of discourse”. We only hope that this research, relying on logic analysis, could contribute to the content related dimension of language games understanding.

REFERENCES

Artigue, M., & Robinet, J. (1982). Conceptions du cercle chez des enfants de l'école élémentaire. *Recherches en didactique des mathématiques*, *3*(2), 5-64.

Barrier, T. (2011). Les pratiques langagières des étudiants en analyse réelle. *Recherches en Didactique des Mathématiques*, *31*(3), 259–290.

Barrier, T., De Vittori, T., & Mathé, A.-C. (2012). Des séances ordinaries comportant une dimension historique. Quels enseignements? *Petit x*, 90.

Bernié, J.-P. (2002). L’approche des pratiques langagières scolaires à travers la notion de « communauté discursive»: Un apport à la didactique comparée? *Revue française de pédagogie*, *141*, 77-88.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Bulf, C., Mathé, A.-C., & Mithalal, J. (2011). Language in geometry classroom. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 649-659), Rzeszów, Poland: ERME.

Duval, R., & Godin, M. (2006). Les changements de regard nécessaires sur les figures. *Grand N*, *76*, 7-27.

Hache, C. (in press). Langage mathématique à la transition primaire/collège. *Actes du colloque de la COPIRELEM 2012*, Quimper, France.

Hintikka, J. (1996). *The principles of mathematics revisited*. Cambridge : Cambridge University Press.

Jaubert, M., Rebière, M., & Bernié, J.-P. (2003). L'hypothèse «communauté discursive»: D'où vient-elle ? Où va-t-elle ? *Cahiers Théodile*, *4*, 51-80.

Kieran, C., Forman, E., & Sfard, A. (2001). Learning discourse: Sociocultural approaches to research in mathematics education. *Educational Studies in Mathematics*, *46*, 1-12.

Mathé, A.-C. (2012). Jeux et enjeux de langage dans la construction de references partagées en classe de géométrie. *Recherches en Didactique des Mathématiques*, *32*(2), 195-228.

Moschkovich, J. (2007). Examining mathematical discourse practices. *For the Learning of Mathematics*, *27*(1), 24-30.

Perrin-Glorian, M.-J., Mathé, A.-C., & Leclerc, R. (2013). Comment peut-on penser la continuité de l'enseignement de la géométrie de 6 à 15 ans ? Le jeu sur les supports et les instruments. *Repères-IREM*, 90.

Sfard, A. (2001). There is more to discourse than meets the ears: Learning from mathematical communication things that we have not known before. *Educational Studies in Mathematics*, *46*, 13-57.

Sfard, A. (2013). Almost 20 years after: Developments in research on language and mathematics. Review of J. N. Moschkovich (Ed.) (2010). *Language and mathematics education: Multiple perspectives and directions for research*. *Educational Studies in Mathematics, 82,* 331-339.

Wittgenstein, L. (1953). *Philosophical investigations* (G. E. M. Anscombe, Trans.). Oxford: Basil Blackwell.

ANNEX

Construction program and example of the expected figure

Stretch out a rope the length of which must correspond to the side of the square to be constructed.
On the ground, mark its ends O and E and its midpoint I.
Plot the circle with a diameter [OE] and circles with radii [OE].
These two large circles are crossed in U and V.
Stretch out a rope between U and V.
Mark as N and S its intersection points with the small circle.
Points U, N, I, S and V are aligned in this order.
Plot the circles with respective centres E, O, N and S the radius of which should measure half the EO length.
These four large circles are crossed two by two in A, B, C and D.
These four points are the vertices of the square.

Follow these instructions and construct a square using the artefacts given to you.